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On moderate injection into a separated supersonic boundary layer, with reattachment

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We consider the problem of mass injection from a flat plate into uniform supersonic flow when the boundary layer is blown off at the leading edge to form a thin free shear layer separating the inviscid injectant layer from the external flow. The injection is moderate, i.e. of the same order of magnitude as the velocity of entrainment into the base of the shear layer. For simplicity, we consider similarity blowing, proportional to $x^{*-\frac{1}{2}}$, where x^* is measured along the plate from the leading edge. We find that the solution in the injectant region, based on initial conditions at the leading edge, is non-unique unless fixed by downstream conditions. When injection is cut off at a finite distance along the plate, this enables us to find a solution to the problem in which the shear layer eventually reattaches to the plate and for which the pressure and the height of the injectant layer are continuous at cut-off. The study provides a partial connexion between the earlier studies of weak blowing in which the boundary layer is not blown off and of strong blowing for which the boundary layer is blown off and the entrainment into the shear layer is negligible.

1. Introduction

We consider a semi-infinite flat plate immersed in a viscous fluid which has velocity U_∞^* in a direction parallel to the plate at large distances upstream. The representative Reynolds number R, defined below in § 2, is taken to be large. A second fluid is injected across a portion of the plate with velocity V_w^* in a direction normal to the mainstream.

The earliest studies of the properties of this fluid motion were concerned with weak blowing in which $V_w^* = O(U_\infty^* R^{-\frac{1}{2}})$ and separation does not occur or with moderate blowing in which V_w^* has the same order of magnitude but separation does occur. Schlichting & Bussman (1943), Iglisch & Gröhne (1945) and Emmons & Leigh (1953) assumed that $V_w^* \propto x^{*-\frac{1}{2}}$, where x^* measures distance along the plate from the leading edge. The governing equations then reduce to a Blasius form and they found that solutions can be found only if the injection rate is less than a certain critical value V_c^*. Emmons & Leigh noticed that this value is identical with the asymptotic entrainment of a free shear layer separating a uniform stream from a fluid at rest. Consequently they suggested that if $V_w^* > V_c^*$ the boundary layer is blown off the plate in the neighbourhood of the leading edge to form a free shear layer. This description is consistent with that already given by Pretsch (1944) for blowing in the presence of a favourable pressure gradient. He showed that, when $V_w^* R^{\frac{1}{2}}/U_\infty^*$ is large and such that a similarity solution of the boundary-layer equations is possible, a very thin shear layer develops which separates the mainstream from the injected fluid. The injectant layer is an order of magnitude thicker than the shear layer above it, but its thickness is still small in comparison with the lateral dimension of the plate and the fluid motion in it is controlled by inviscid forces only. A crucial feature of Pretsch's solution is the favourable pressure gradient, which serves to drive the injectant downstream, and it fails when this gradient vanishes.

Catherall, Stewartson & Williams (1965) have examined the response of the boundary layer to weak uniform injection when the imposed pressure gradient is zero. The boundary layer takes on a non-similar form near the leading edge and separates at a finite distance from the leading edge. Near this point the reduced skin friction τ takes the form

$$\tau \propto \left(\frac{x_s^* - x^*}{x_s^* \log[(x_s^* - x^*)/x_s^*]}\right)^2 \tag{1.1}$$

and the displacement thickness $\simeq x_s^* R^{-\frac{1}{2}} \log \tau^{-1}$ (Brown & Stewartson 1969). Presumably the boundary layer is then blown off the plate somewhat in the manner discussed by Emmons & Leigh but the precise mechanism is still not clear and may even involve hysteresis (Smith & Stewartson 1973*b*).

Cole & Aroesty (1968) initiated the study of moderate blowing in a supersonic stream and took the injection to be strong enough to allow neglect of the entrainment into the shear layer by comparison. They assumed that the injectant region is inviscid and determined the relation between its thickness, as defined by the position of the much thinner shear layer above it, and the injection velocity by means of a formula similar to the one given by Gadd, Jones & Watson (1963). Klemp & Acrivos (1972) made a parallel study for an incompressible fluid and found that when $V_w^* \propto x^{*-\frac{1}{2}}$ for all $x^* > 0$ there is a double-structured solution for large x^* analogous to that found by Cole & Aroesty for supersonic flow. Amr & Kassoy (1973) further investigated the structure of the supersonic flow field when entrainment may not be neglected.

Since there must be a favourable pressure gradient above the separated boundary layer to drive the injectant downstream, there must also be a pressure rise upstream of separation or downstream of any reattachment to ensure that conditions at infinity are satisfied. The first calculation of a complete pressure field was made by Smith & Stewartson (1973*a*) for strong blowing in which $V_w^*/U_\infty^* \sim R^{-\frac{3}{8}}$, so that entrainment can be neglected, and in which separation takes place ahead of the onset of blowing. They found that separation occurs through a free interaction and that the pressure rise has reached a plateau before blowing begins. The pressure then decreases throughout the blowing region and is assumed to return to its undisturbed value at the cut-off of injection; this condition ensures a unique solution. Later Stewartson (1974*b*) analysed the transition regions ahead of blowing and at cut-off and demonstrated that the discontinuities in the theory may legitimately be smoothed out.

The connexions between the theory for moderate blowing over an infinite length and that for weak blowing leading to separation or, on the other hand, for strong blowing remain imperfectly understood. Amr & Kassoy (1976) initiated the study of moderate blowing over a finite length of the plate when the mainstream is supersonic and the injection is proportional to $x^{*-\frac{1}{2}}$. They based their theory on the premise that disturbances can only travel downstream because of the hyperbolic or parabolic character of the governing equations. Upstream of cut-off they therefore used the similarity solutions which they had earlier discussed for an infinite length of blowing. Beyond cut-off, the free shear layer continues to develop, entraining fluid normally, but there is no longer a supply from the plate. Consequently they assumed that re-attachment of the shear layer to the plate occurs when all the injectant has been absorbed. On computing the flow field downstream of cut-off satisfying this condition, they found discontinuities in both the displacement thickness and the pressure at the cut-off point. They reasoned that a narrow transition region exists at this point to smooth out these jumps.

Now, discontinuities of this type have not been found before in boundary-layer theory and even a pressure jump seems to take place only in special circumstances. In the present class of problems the pressure jump at the onset of blowing can be justified on the grounds that a compressive free interaction has taken place, possibly with a local region of reversed flow. The pressure jump at reattachment may yield to analysis eventually even though it is imperfectly understood at present. It has been described in a series of papers by Chang & Messiter (1968), Burggraf (1970, 1973, 1975) and Messiter, Hough & Feo (1973) on the basis that it is largely inviscid in character, the pressure and inertia terms becoming large in comparison with the viscous terms. The pressure jump is then needed to turn the fluid in the shear layer round to continue the recirculating motion which precedes reattachment. Comparable theories describing the discontinuities at cut-off do not appear to be feasible so in this paper we re-examine the whole problem of moderate blowing over a finite length of the plate to find an alternative solution free of this anomaly.

We find that the solution upstream of cut-off contains an arbitrary constant and is therefore not independent of conditions downstream as was assumed by Amr & Kassoy (1976). This constant may be interpreted as an effective origin shift (see (4.15) below) and as such its role is similar to the origin shift in the free-interaction problem (Stewartson 1974*a*). With this constant at our disposal it is possible to ensure the continuity of the displacement thickness and the pressure at cut-off and thus a satisfactory completion of the description of the flow field. As the intensity of blowing increases, we find that the predictions of our model are consistent with those of Smith & Stewartson (1973*a*) for strong blowing, particularly with respect to the pressure at cut-off, although the differences in the injection velocity distribution preclude a full comparison. A relation with the theory for weak blowing (e.g. Catherall *et al.* 1965) also seems likely but further study is needed to settle this point.

2. Formulation

A viscous incompressible fluid flows past a semi-infinite plate defined in Cartesian co-ordinates (x^*, y^*) by $0 \leqslant x^* < \infty$, $y^* = 0$. At large distances from the plate the fluid is in uniform supersonic motion parallel to the x^* axis with velocity U^*_∞. At a general point let ρ^* be the density of the fluid, p^* the pressure, T^* the absolute temperature and (u^*, v^*) the components of velocity, also let μ^* be the coefficient of viscosity and let it be proportional to T^*. A second fluid, also viscous and compressible, is injected across the plate with velocity $(0, V^*_w)$. In the case when

$$V^*_w = U^*_\infty \frac{T^*_w}{T^*_\infty} C \left(\frac{\mu^*_\infty}{2\rho^*_\infty U^*_\infty x^*}\right)^{\frac{1}{2}}, \quad 0 \leqslant x^* < \infty, \tag{2.1}$$

where C is a constant and the suffixes ∞ and w refer to conditions at large distances from the plate and on the plate respectively, Iglisch & Gröhne (1945) have shown that once the momentum equation of the boundary layer has been reduced to the Blasius form by the use of the Howarth–Dorodnytsin variables a solution can be found if

$$C \leqslant C_0 = 0{\cdot}87574, \tag{2.2}$$

the numerical value being due to Emmons & Leigh (1953). When $C > C_0$ no solution was found and they suggested that the boundary layer is then blown off the plate in some sense and forms a free shear layer. Such a possibility had already been investigated for boundary layers in a favourable pressure gradient by Pretsch (1944). Emmons & Leigh (1953) then showed that for a free shear layer in which $u^* \to U^*_\infty$ as $y^* \to \infty$ and $u^* \to 0$ as $y^* \to -\infty$ the solution of the Blasius equation gives

$$\lim_{y^* \to -\infty} v^* = U^*_\infty \frac{T^*_w}{T^*_\infty} C_0 \left(\frac{\mu^*_\infty}{2\rho^*_\infty U^*_\infty x^*}\right)^{\frac{1}{2}}, \tag{2.3}$$

which defines the rate at which fluid is entrained into the shear layer once similarity is established. They left open the question of the position of the shear layer and the mode of adjustment of the value of v^* at the wall to the value (2.3) at the base of the shear layer. These points were answered by Cole & Aroesty (1968) for $C \gg 1$ and by Amr & Kassoy (1973) for $C = O(1)$, both studies finding a similarity solution for the injectant region.

In this paper we consider injection to occur only over a finite length L^* of the plate so that

$$V^*_w = \begin{cases} U^*_\infty \dfrac{T^*_w}{T^*_\infty} C \left(\dfrac{\mu^*_\infty}{2\rho^*_\infty U^*_\infty x^*}\right)^{\frac{1}{2}}, \quad 0 \leqslant x^* < L^*, & (2.4a) \\ 0, \quad x^* > L^*, & (2.4b) \end{cases}$$

and we assume that $R = U^*_\infty L^* \rho^*_\infty / \mu^*_\infty \gg 1$ so that the boundary-layer arguments are appropriate. We suppose that over a region $0 \leqslant x^* \leqslant x^*_R$, where $x^*_R > L^*$, the oncoming fluid is separated from the injected fluid by a self-similar free shear layer of thickness $O(R^{-\frac{1}{2}}L^*)$. The injected fluid is confined within a distance $\delta^*(x^*)$ of the plate, where

$$L^* R^{-\frac{1}{2}} \ll \delta^*(x^*) \ll L^*. \tag{2.5}$$

Subsequently, the free shear layer reattaches to the plate and a conventional boundary layer develops. A sketch of the flow is shown in figure 1. Here

$$x^*_F = (C/C_0)^2 L^*, \tag{2.6}$$

which is the value of x^* at which all the injected fluid has been entrained into the shear layer. Either the shear layer reattaches at this point and $x^*_R = x^*_F$ or it continues to entrain fluid which comes from a length of reversed flow in $x^*_F < x^* < x^*_R$. Of course, it may never reattach, in which case $x^*_R \to \infty$. Later we discuss these possibilities and show that in fact $x^*_R - x^*_F \ll 1$.

To find a solution to the problem we require two relations between p^* and δ^*. The first is derived from the equations of motion in the injectant region. Here the injected fluid may be presumed to have constant temperature and density to lowest order, as

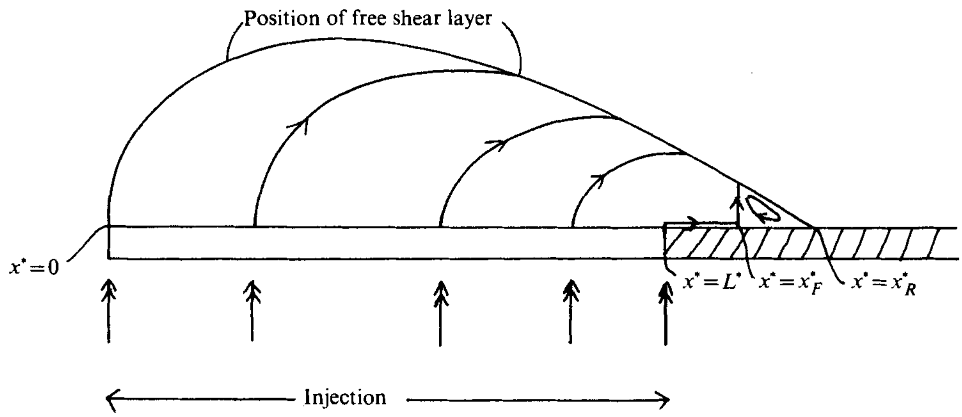

FIGURE 1. Flow in the injectant layer.

Amr & Kassoy (1973) have demonstrated. We adopt the following non-dimensional variables:

$$\left.\begin{aligned} x^* &= L^*x, \quad y^* = L^*R^{-\frac{1}{3}}(M^{*2}_\infty-1)^{\frac{1}{6}}(T^*_w/T^*_\infty)^{\frac{1}{3}}C^{\frac{2}{3}}y,\\ u^* &= U^*_\infty R^{-\frac{1}{6}}(M^{*2}_\infty-1)^{-\frac{1}{6}}(T^*_w/T^*_\infty)^{\frac{2}{3}}C^{\frac{1}{3}}u, \quad v^* = U^*_\infty R^{-\frac{1}{2}}(T^*_w/T^*_\infty)Cv,\\ p^* &= p^*_\infty+\rho^*_\infty U^{*2}_\infty R^{-\frac{1}{3}}(M^{*2}_\infty-1)^{-\frac{1}{3}}(T^*_w/T^*_\infty)^{\frac{1}{3}}C^{\frac{2}{3}}p(x),\\ \delta^* &= L^*R^{-\frac{1}{3}}(M^{*2}_\infty-1)^{\frac{1}{6}}(T^*_w/T^*_\infty)^{\frac{1}{3}}C^{\frac{2}{3}}\delta(x), \end{aligned}\right\} \tag{2.7}$$

where M^*_∞ is the Mach number. These are similar to those used by Amr & Kassoy (1976), the only difference being that ours are also scaled with respect to C. The equations of motion in the injectant layer now become

$$u_x+v_y = 0, \quad uu_x+vu_y = -p_x. \tag{2.8}$$

In the same way that Cole & Aroesty found that there is an integral relation between p and δ for strong blowing, we find that in the weak blowing problem which we are considering

$$\delta(x) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\alpha^2 x}^{x} \frac{t^{-\frac{1}{2}}\,dt}{[(p(t)-p(x))]^{\frac{1}{2}}}, & 0 < x < 1, \qquad (2.9a)\\[2ex] \dfrac{1}{2}\displaystyle\int_{\alpha^2 x}^{1} \frac{t^{-\frac{1}{2}}\,dt}{[(p(t)-p(x))]^{\frac{1}{2}}}, & 1 < x < x_F, \qquad (2.9b) \end{cases}$$

where $\alpha^2 = (C_0/C)^2$.

A further relation between p and δ emerges from the application of slender-body theory in the external flow, namely

$$p(x) = \delta'(x). \tag{2.10}$$

Two conditions on $\delta(x)$ are now necessary to solve the integro-differential equations (2.9) and (2.10). The boundary layer separates close to the leading edge so

$$\delta(0) = 0. \tag{2.11}$$

The second condition is the value of δ at x_F, which may be found by considering the reversed flow region. We shall show that

$$\delta(x_F) = 0. \tag{2.12}$$

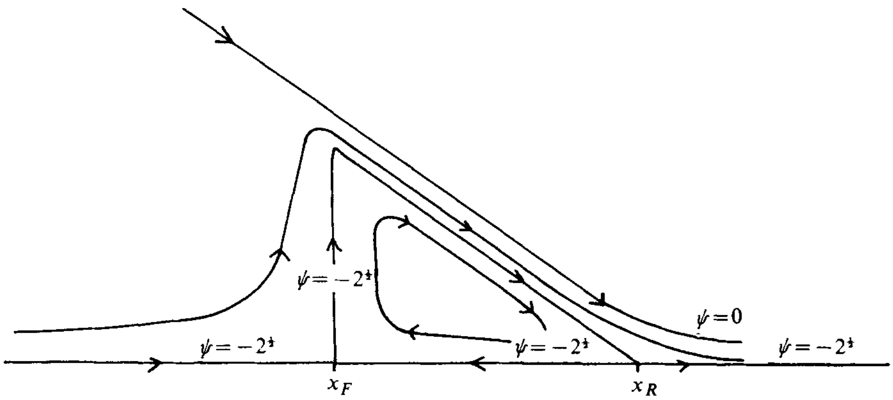

FIGURE 2. Flow near reattachment.

If the shear layer does not reattach, $x_R \to \infty$. The injectant increases the total flux past the plate by a factor $1 + O(R^{-\frac{1}{2}})$ so, in the limit $x \to \infty$, p^* can be changed from its original value p^*_∞ by only a factor $1 + O(R^{-\frac{1}{2}})$. We therefore assume that $p(x) \to 0$ as $x \to \infty$. Consider $x \gg 1$. The entrainment by the shear layer must be fed by fluid coming from $x \sim \infty$, where $u < 0$, and this balance requires

$$u\delta(x) \sim x^{\frac{1}{2}}. \tag{2.13}$$

If $uu_x \sim p_x$, then $\delta(x) = Bx^{\frac{2}{3}}$, where B is some constant, and $u \sim x^{-\frac{1}{6}}$, so that $u \to 0$ as $x \to \infty$. This form is however contradictory because the pressure gradient $\delta''(x)$ is then favourable and cannot sustain a reversed flow. An alternative is that $u \sim 1$, $x \gg 1$ and $p_x \ll uu_x$. If this is so (2.13) implies $\delta(x) = Bx^{\frac{1}{2}}$. The favourable pressure gradient is now consistent because it will simply retard the reversed flow. The finite velocity when $x \gg 1$ must however be set up by some external mechanism. Although this is possible in principle, it is outside the scope of this paper. We conclude that $x_R - x_F$ must be finite. Figure 2 shows the flow pattern in $x_F < x < x_R$ for this case. The limiting streamline $\psi = -2^{\frac{1}{2}}$, which divides the injectant from the reversed flow, is entrained by the shear layer at $x_F = 1/\alpha^2$. At x_R it strikes the plate at an angle and divides. The fluid below it is turned back and feeds the reversed flow while the fluid above continues along the plate to re-establish the boundary layer past x_R.

Because the pressure gradient and acceleration are large near x_R, viscous forces are unimportant in the major part of the flow here. We therefore look for an inviscid reattachment region at x_R of the form discussed by Messiter *et al.* (1973).

We have shown that p^* is ultimately changed from p^*_∞ by only a factor $1 + O(R^{-\frac{1}{2}})$. At x_R there is therefore a sudden rise in the pressure from $p(x_R-)$ to its undisturbed value $p(x_R+) = 0$. This must be strong enough to turn the fluid under the streamline $\psi = -2^{\frac{1}{2}}$ back into the reversed flow region. However, $p(x_R-)$ must still be consistent with $p(x)$ upstream and cannot be greater than $O(1)$. Fluid on $\psi = -2^{\frac{1}{2}}$ is accelerated in the shear layer after its entrainment at x_F and if $x_R - x_F$ were $O(1)$ it would have reached the same order of velocity as the external flow before reattachment. In this case $p(x_R-) = O(R^{\frac{1}{4}})$ ($|p^* - p^*_\infty| = O(\rho^*_\infty U^{*2}_\infty)$), which is contradictory, and we infer that $x_R - x_F \ll 1$. In this event

$$\delta(x_F) = O((x_R - x_F)\,p(x_R-)) \ll 1. \tag{2.14}$$

A more detailed examination of the reattachment region reveals that $\delta(x_F)$ cannot be greater than $O(R^{-\frac{1}{2}})$ (Diver 1978), on which scale viscous forces come into play. The assumption of inviscid attachment is therefore formally incorrect, but we may still use the condition (2.12) that $\delta(x_F) = 0$, which is all that we require to complete the problem posed in (2.9)–(2.11).

A general feature of the solution is that the pressure gradient is zero immediately after cut-off, i.e.

$$p'(1+) = 0. \tag{2.15}$$

The proof of this result will be given elsewhere (Diver 1978) but is based on the fact that $p'(1+) \neq 0$ implies $u \propto (x-1)^{\frac{1}{2}}$ as $x \to 1+$ if $y = 0$. Hence the associated form of δ has a like singularity which in turn implies an infinite pressure in virtue of (2.10), thus contradicting the assumption of a finite pressure gradient at $x = 1+$.

3. Marginal blow-off

We define marginal blow-off by the condition $0 < 1-\alpha \ll 1$ ($\alpha = C_0/C$). Thus the normal velocity on the plate is only just greater than that at the base of the shear layer, so δ, the reduced height of the injectant region, is small and reattachment takes place soon after cut-off. The presence of a small parameter enables us to find the solution analytically and from this solution we can infer the behaviour of the flow for general values of α.

Since the range of the integral in (2.9*a*) is small $[O(1-\alpha)]$ it may be simplified by using the formula

$$p(t) = p(x) + (t-x)p'(x) + O((1-\alpha)^2), \quad \alpha^2 x \leqslant t \leqslant x. \tag{3.1}$$

On making use of (2.10), the integral equation then reduces to

$$\Delta(x)^2 \Delta''(x) = -2 + O(1-\alpha), \tag{3.2}$$

where

$$\Delta(x) = (1-\alpha)^{-\frac{1}{3}} \delta(x), \tag{3.3}$$

which confirms the earlier remark that δ is small. We note that Kassoy (1974) has also discussed the marginal blow-off problem, but for an infinite length of injection. He found when $1-\alpha = O(R^{-\frac{1}{2}})$ that $\delta^* = O(L^* R^{-\frac{1}{6}})$, which is of the same order of magnitude as the thickness of the shear layer. This result is formally consistent with (3.3), which implies that $\delta^* = O(L^* R^{-\frac{1}{3}}(1-\alpha)^{\frac{1}{3}}) = O(L^* R^{-\frac{1}{2}})$ when $1-\alpha = O(R^{-\frac{1}{2}})$.

Integration of (3.2) gives

$$\Delta'(x)^2 - 4/\Delta(x) = -4/A \tag{3.4}$$

and a further integration shows that there are three branches to the solution, corresponding respectively to $1/A = 0$, $A < 0$ and $A > 0$.

If $1/A = 0$, (3.4) becomes

$$\Delta(x) = (3x)^{\frac{2}{3}}, \tag{3.5}$$

which is the similarity solution derived by Amr & Kassoy (1973) for an infinite length of blowing. Here $\Delta'(x)$ is positive for all finite x and tends to zero only if (3.5) holds in $0 < x < \infty$ and x tends to infinity.

If $A < 0$, (3.4) becomes

$$x = \frac{|A|^{\frac{3}{2}}}{2}\left\{\left[\frac{\Delta(x)}{|A|}\left(1+\frac{\Delta(x)}{|A|}\right)\right]^{\frac{1}{2}} - \log\left[\left(\frac{\Delta(x)}{|A|}\right)^{\frac{1}{2}} + \left(1+\frac{\Delta(x)}{|A|}\right)^{\frac{1}{2}}\right]\right\}. \tag{3.6}$$

When this holds in the full range $0 < x < \infty$, $\Delta'(x)$ tends to the finite limit $|A|^{\frac{1}{2}}$ and $\Delta(x)$ is linear as x tends to infinity.

Finally, if $A > 0$, (3.4) becomes

$$\left(\frac{\Delta(x)}{A}\right)^{\frac{1}{2}} = \sin\left\{\frac{2x}{A^{\frac{3}{2}}} + \left[\frac{\Delta(x)}{A}\left(1-\frac{\Delta(x)}{A}\right)\right]^{\frac{1}{2}}\right\}. \tag{3.7}$$

This solution terminates at $x_s = \frac{1}{2}\pi A^{\frac{3}{2}}$, near which point

$$\Delta(x) = 3^{\frac{2}{3}}(x-x_s)^{\frac{2}{3}} - (3^{\frac{4}{3}}/5A)\,(x-x_s)^{\frac{4}{3}} + O((x-x_s)^2). \tag{3.8}$$

When $x > \frac{1}{4}\pi A^{\frac{3}{2}}$, $\Delta'(x) < 0$ and the shear layer turns down towards the plate, but because more fluid is being injected than is being entrained, it is unable to reattach smoothly. Thus the solution terminates in the singularity shown by (3.8).

We now see that there are essentially only three solutions of (3.2): the similarity form with $1/A = 0$ and two others of the form

$$x^{-\frac{2}{3}}\delta(x) = F_1(x/A^{\frac{2}{3}}) \quad \text{and} \quad x^{-\frac{2}{3}}\delta(x) = F_2(x/|A|^{\frac{2}{3}}),$$

depending on the sign of A. For example, if $A > 0$, (3.7) may be written as

$$[G_1(z)]^{\frac{1}{2}} = \sin\{2z + [G_1(z)\,(1-G_1(z))]^{\frac{1}{2}}\}, \tag{3.9}$$

where

$$G_1(z) = z^{\frac{2}{3}}F_1(z), \quad z = x/A^{\frac{2}{3}}. \tag{3.10}$$

Which of these three solutions we choose is determined by the matching with the solution when $x > 1$. Thus, despite the fact that disturbances only travel downstream in supersonic flow, there is an inbuilt mechanism which enables the flow upstream to adjust to conditions beyond cut-off. This situation is similar to the free interaction in the boundary layer which occurs ahead of a phenomenon such as an incident shock or convex corner. Here there are also three ways in which the evolving boundary layer can behave. If it does not continue to take the undisturbed Blasius form in which the pressure is zero, either the pressure rises and the boundary layer separates, after which the pressure reaches the constant plateau value, or the pressure falls and the solution terminates in a singularity with $p \to -\infty$. In the blowing problem, the pressure is not uniformly zero for the central branch (3.5), but the behaviour of the pressure at large values of x in the solutions (3.5)–(3.7) is otherwise analogous to that in the interaction problem.

However, (3.5)–(3.7) hold only in $0 < x < 1$ and we must consider the flow beyond $x = 1$. In the marginal blow-off problem the length between cut-off and reattachment is small $[O(1-\alpha)]$ and the pressure there must be large if it is able to reduce $\Delta(x)$ to zero at x_R. Since we wish both $\Delta(x)$ and $\Delta'(x)$ to be continuous near $x = 1$, it follows that

$$\Delta'(1) = O((1-\alpha)^{-1}\Delta(1)). \tag{3.11}$$

There is now a contradiction if $\Delta(1) \neq 0$ because then (3.4) cannot be satisfied, so

$$\Delta(1) = 0. \tag{3.12}$$

α	0·97	0·98	0·99
$D'(0)$	$-1{\cdot}117$	$-1{\cdot}169$	$-1{\cdot}200$

TABLE 1

This condition is sufficient to determine the solution uniquely. Of the three branches (3.5)–(3.7), (3.5) and (3.6) both have $\Delta(1)$, $\Delta'(1) > 0$ and cannot satisfy (3.12). Only (3.7) is appropriate, with A determined by (3.12), which gives

$$A = (2/\pi)^{\frac{2}{3}}. \tag{3.13}$$

The solution for $\Delta(x)$ in $0 < x < 1$ is now complete to lowest order, but the behaviour of $\Delta(x)$ in $1 < x < x_R$ and the values of the pressure at cut-off and reattachment are as yet unknown. We therefore continue by examining the region $|1-x| = O(1-\alpha)$ in more detail. Here we write

$$x = 1+(1-\alpha)z, \quad \delta(x) = (1-\alpha)D(z), \tag{3.14}$$

where the scale for $\delta(x)$ is suggested by (3.4) and (3.11). Equations (2.9) and (2.10) then reduce to the following integro-differential equation for $D(z)$ in $-\infty < z < 2$:

$$D(z) = \frac{1}{2}\int_{z-2}^{\min(0,z)} \frac{ds}{(D'(s)-D'(z))^{\frac{1}{2}}}. \tag{3.15}$$

The range of integration is finite for general values of z and we cannot simplify the equation by linearizing $D'(s)$ as we did for (2.9a). Thus its solution seems to require numerical study.

Near reattachment however, where $|z-2| \ll 1$, we may write

$$D'(s) = D'(0)+O((z-2)^2) \tag{3.16}$$

by virtue of (2.15) and this reduces (3.15) to

$$D(z)\,(D'(0)-D'(z)) = -\tfrac{1}{2}(z-2)+O((z-2)^2), \quad |z-2| \ll 1. \tag{3.17}$$

The expansion of $D(z)$ in powers of $z-2$ must have the leading term $D'(2)(z-2)$ because $D(2) = 0$. Thus, setting the coefficient of $z-2$ to zero in (3.17), we find the following relation between the pressure $D'(0)$ at cut-off and the pressure $D'(2)$ at reattachment:

$$D'(2)\,(D'(0)-D'(2))^{\frac{1}{2}} = -\tfrac{1}{2}. \tag{3.18}$$

As an alternative to solving (3.15) numerically to find $D'(0)$, we may make use of the results available from the general procedure to be described in the next section. In table 1 we display the values of $D'(0)$ for a range of values of $1-\alpha$ near zero.

It now seems reasonable to conjecture that $d(D'(0))/d\alpha = 0$ when $\alpha = 1$; on this basis

$$D'(0) = -1{\cdot}210 \tag{3.19}$$

and then, by the use of (3.18),

$$D'(2) = -1{\cdot}348. \tag{3.20}$$

The numerical results over the same range of values of α also confirm that $\Delta(x)$ is linear near $x = x_R$, as was assumed in deriving (3.18), and show that this is true over the whole range $1 < x < x_R$.

Thus the pressure generally appears to be continuous as $C \to C_0 +$ with that derived from the earlier solutions obtained when $C < C_0 +$; for the present study

$$p = O(C - C_0)^{\frac{1}{3}} \quad \text{as} \quad C \to C_0 \tag{3.21}$$

while without blow-off $p^* - p^*_\infty$ is formally $O(R^{-\frac{1}{2}})$ and hence from (2.7)

$$p = 0 \quad \text{if} \quad C < C_0. \tag{3.22}$$

However, in the neighbourhood of reattachment the situation is different for we now see that when $(x_R - x) = O(1-\alpha)$, p tends to a finite limit as $C \to C_0 +$. One may expect therefore that in practice a small but rapid rise occurs near the cut-off point as the blowing rate passes through the critical level for blow-off.

4. The numerical procedure for $0 < \alpha < 1$

If $0 < \alpha < 1$, we may still treat (2.9a) analytically for $x \ll 1$. We assume that the expansion of δ in ascending powers of x takes the form

$$\left.\begin{aligned} \delta(x) &= ax^{\frac{2}{3}} - bx^m + \ldots, \quad x \ll 1, \\ a &= [\tfrac{3}{2}(2+\alpha^{\frac{2}{3}})^2 (1-\alpha^{\frac{2}{3}})]^{\frac{1}{3}}, \end{aligned}\right\} \tag{4.1}$$

where b and m are functions of α to be determined. The first term of (4.1) is the similarity solution found by Amr & Kassoy (1973) for an infinite length of blowing and is by itself a solution to (2.9a). However, on the basis of the previous section, we expect that the solution to (2.9a), (2.10) and (2.11) is non-unique and so look for a non-zero second term in (4.1). This exists for all $b \neq 0$ provided that m satisfies the transcendental equation

$$\frac{8(2+\alpha^{\frac{2}{3}})(1-\alpha^{\frac{2}{3}})^{\frac{1}{2}}}{3m} = \int_{\alpha^2}^{1} \frac{(u^{m-1}-1)\,du}{(1-u^{\frac{1}{3}})^{\frac{3}{2}}}, \tag{4.2}$$

which follows if we substitute (4.1) in (2.9a) and equate the coefficient of x^m to zero.

In the limit $\alpha = 1$, (4.2) reduces to

$$(m - \tfrac{4}{3})(m + \tfrac{1}{3}) = 0, \tag{4.3}$$

and of the two possible solutions $m = \frac{4}{3}$ is relevant to (4.1) since we are assuming an expansion in ascending powers of x. From § 3 we see that the expansion (4.1) continues

$$(1-\alpha)^{-\frac{1}{3}}\,\delta(x) = 3^{\frac{2}{3}}x^{\frac{2}{3}} - \frac{3^{\frac{4}{3}}}{5A}x^{\frac{4}{3}} - \frac{3^3x^2}{175A^2} + \ldots \tag{4.4}$$

if $1-\alpha \ll 1$, which is an expression of the fact that $x^{-\frac{2}{3}}\delta(x)$ has the form $F(x/|A|^{\frac{3}{2}}, \alpha)$ as has already been pointed out. For general values of α, (4.2) may be solved by Newton–Raphson iteration and in table 2 we display the values of m corresponding to various values of α in the range $0 \leqslant \alpha \leqslant 1$. It is observed that m is always close to $\frac{4}{3}$. The constant b in (4.1) plays the same role as A in (3.4) and its sign distinguishes between two branches of the solution. Anticipating the numerical results to be discussed below we note that, if $b < 0$, p tends to a finite limit as $x \to \infty$ and $x^{-\frac{2}{3}}\delta(x)$ takes the form $F_1(|b|x^{m-\frac{2}{3}}, \alpha)$. On the other hand, if $b > 0$, the solution terminates at a finite value x_T of x; for $x < x_T$, $x^{-\frac{2}{3}}\delta(x)$ takes the form $F_2(bx^{m-\frac{2}{3}}, \alpha)$ and $p \to -\infty$ as $x \to x_T -$.

α	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
m	1·403	1·397	1·390	1·382	1·375	1·367	1·359	1·353	1·346	1·339	1·333

TABLE 2

The second solution of (4.3) corresponds to an eigenvalue in the asymptotic solution for δ as $x \to \infty$ and would be needed if there were blowing across the plate for all $x > 0$, the injection velocity being asymptotically but not identically proportional to $x^{*-\frac{1}{2}}$. It is noted that this eigenvalue is a decreasing function of α and that

$$m \log(1/\alpha) \to -\tfrac{1}{2}\log\tfrac{16}{3} \quad \text{as} \quad \alpha \to 0.$$

Before embarking on the numerical solution, it is observed that near reattachment, i.e. when $|x - x_R| \ll 1$, (2.9b) may be simplified in a similar way to that used to derive (3.18). We find, by expanding δ in powers of $x_R - x$, that δ is linear to lowest order and

$$p(x_R)\,(p(1) - p(x_R))^{\frac{1}{2}} = -\tfrac{1}{2}\alpha^2, \tag{4.5}$$

a result that proves useful in deciding the correct choice of b.

A direct numerical solution of the integral equation (2.9a), by forward integration of x, presents difficulties because the procedure converges very slowly, if at all, probably owing to the basic non-uniqueness of the solution in the absence of a downstream condition, manifested above by b. Consequently it was decided to revert to (2.8) together with the slender-body condition (2.10) in the form

$$\tfrac{1}{2}ff_{\eta\eta} + \left(\frac{1}{6} + \frac{d\theta}{dt}\right)f_\eta^2 + e^{3\theta}\left(-\frac{2}{9} + \frac{1}{3}\frac{d\theta}{dt} + \left(\frac{d\theta}{dt}\right)^2 + \frac{d^2\theta}{dt^2}\right) = f_\eta f_{\eta t} - f_{\eta\eta} f_t, \tag{4.6}$$

where

$$\psi = x^{\frac{1}{2}} f(x, \eta), \quad \eta = y/\delta(x), \tag{4.7}$$

$$\delta(x) = x^{\frac{2}{3}} \exp[\theta(x)], \quad x = \exp(t - t_0) \qquad (4.8), (4.9)$$

and t_0 is a constant to be defined later. The use of η transforms the injectant layer into a rectangle while (4.8) and (4.9) remove the explicit dependence of (2.8) on x or t. The form (4.8) is preferable to writing $\delta(x) = x^{\frac{2}{3}}\theta(x)$ because it makes the coefficients of f_η^2 and the third term in (4.6) simpler to deal with when performing Newton iteration. If, as we shall assume, $x^{-\frac{2}{3}}\delta(x)$ is a function of $bx^{m-\frac{2}{3}}$, (4.9) changes the effect of b into an origin shift of t. Thus there are only three solutions of (4.6), all unique apart from the origin of t; one is the constant similarity solution $x^{-\frac{2}{3}}\delta(x) = a$ while the other two correspond to $b < 0$ and to $b > 0$.

The boundary conditions satisfied by f in $-\infty < t < t_0$ are

$$\left.\begin{aligned} f &= -2^{\frac{1}{2}}, \quad f_\eta = 0, \quad \eta = 0, \\ f &= -\alpha \times 2^{\frac{1}{2}}, \quad \eta = 1. \end{aligned}\right\} \tag{4.10}$$

After cut-off at $t = t_0$, f satisfies the conditions

$$f = \begin{cases} -2^{\frac{1}{2}} \exp[-\frac{1}{2}(t - t_0)], & \eta = 0, \qquad (4.11a) \\ -\alpha \times 2^{\frac{1}{2}}, & \eta = 1, \qquad (4.11b) \end{cases}$$

in $t_0 < t < t_R$; moreover, in this range of t

$$f_\eta^2 = 2e^{2\theta}\left[\left(\frac{2}{3} + \frac{d\theta(t_0)}{dt}\right)\exp\left(\theta(t_0) + t - \frac{1}{3}t_0\right) - \left(\frac{2}{3} + \frac{d\theta}{dt}\right)e^{\theta}\right], \quad \eta = 0, \tag{4.12}$$

which is the condition that $\frac{1}{2}u^2+p$ is constant on the streamline $\psi = -2^{\frac{1}{2}}$. This is used instead of $f_\eta = 0$ on $\eta = 0$ because the momentum equation in (2.8) cannot be satisfied by $u = v = 0$ on $y = 0$ unless $p_x = 0$. As Stewartson (1974*b*) points out, a viscous sublayer is necessary downstream of $x = 1$ to reduce u to zero on the plate.

We solve (4.6) and (4.10)–(4.12) by the box method, which has been described by Keller & Cebeci (1971) and most recently by Cebeci & Bradshaw (1977, p. 213). We step downstream in t and use Newton iteration to solve the difference equations. For $0{\cdot}85 \leqslant \alpha < 1$, 10 steps in η and step lengths of 0·1 or 0·05 in t are sufficient to give convergence to within a tolerance of 10^{-6}. For $\alpha = 0{\cdot}5$, double the number of steps in η are needed since the range of f and f_η in $0 \leqslant \eta \leqslant 1$ is much larger. Beyond cut-off, convergence is slower with these step lengths, but we choose to be satisfied with convergence to within 10^{-3}. We find that, as the solution converges at any t position, it may begin to oscillate as it approaches the correct value. This is thought to be due to the hyperbolic nature of (4.6) and is corrected by using relaxation once the oscillation has been detected.

The initial conditions on f and f_η in $0 \leqslant \eta \leqslant 1$ are found by solving the equation

$$ff_{\eta\eta}+\tfrac{1}{3}f_\eta^2-\tfrac{4}{9}a^3 = 0, \tag{4.13}$$

where a is defined in (4.1), to which (4.6) reduces in the limit $t \to -\infty$. However, by starting with these values and using the correct initial conditions $\theta = \log a$ and $d\theta/dt = 0$ we have still not specified the type of solution since these values must hold at $x = 0$ for all three branches. Nevertheless, the presence of numerical errors means that even after the first step in t, θ and $d\theta/dt$ are not exactly equal to their initial values. Thus the solution begins to diverge from the similarity form which one would otherwise expect and b has effectively been given some value. We find that the solution then follows whichever branch the sign of this b dictates and we cannot recover the unstable case $b = 0$. In order that we may choose the sign of b, we give $d\theta/dt$ a small positive or negative impulse initially: positive for $b < 0$, negative for $b > 0$. Thus instead of beginning the solution at $x = 0$, we start it at $x = x_0$ ($\ll 1$). We define t_0 in (4.9) by

$$x_0 = \exp(-t_0), \tag{4.14}$$

where (4.1) shows that

$$\exp(-t_0) \simeq \left(\frac{3a[d\theta/dt]_{t=0}}{2b}\right)^{1/(m-\frac{2}{3})}. \tag{4.15}$$

The initial values of f, f_η and θ are unchanged to lowest order because $x_0 \ll 1$.

Once the calculations have been performed it becomes clear that for a fixed α the two solution branches $b \gtrless 0$ are unique apart from the origin of t, as we anticipated. Whatever initial value of $d\theta/dt$ is used, the solution remains the same, but is shifted along the t axis. Figure 3 shows an example of each branch and confirms that if $b > 0$, δ is linear and p tends to a finite limit as $x \to \infty$, since $x^{\frac{1}{3}}p(x) \to x^{-\frac{2}{3}}\delta(x)$ as $x \to \infty$. It also shows that the solution terminates in a singularity, as discussed above, if $b > 0$. As in the marginal blow-off problem, we need $b > 0$ for a correct match at $x = 1$. We choose a cut-off point by setting $t = t_0$ at some value $\theta(t_0)$ of θ. We then continue to solve (4.6) subject to the new boundary conditions (4.11) and (4.12) until the reattachment point $t_R = 2\log(1/\alpha)$ is reached, where we require $\theta(t) \to -\infty$ as $t \to t_R$ in order to satisfy $\delta(x_R) = 0$. Since $p(x_R)$, which is given by

$$p(x_R) = \left(\frac{2}{3}+\frac{d\theta}{dt}(t_R)\right)\exp\left(\theta(t_R)-\frac{t_R-t_0}{3}\right),$$

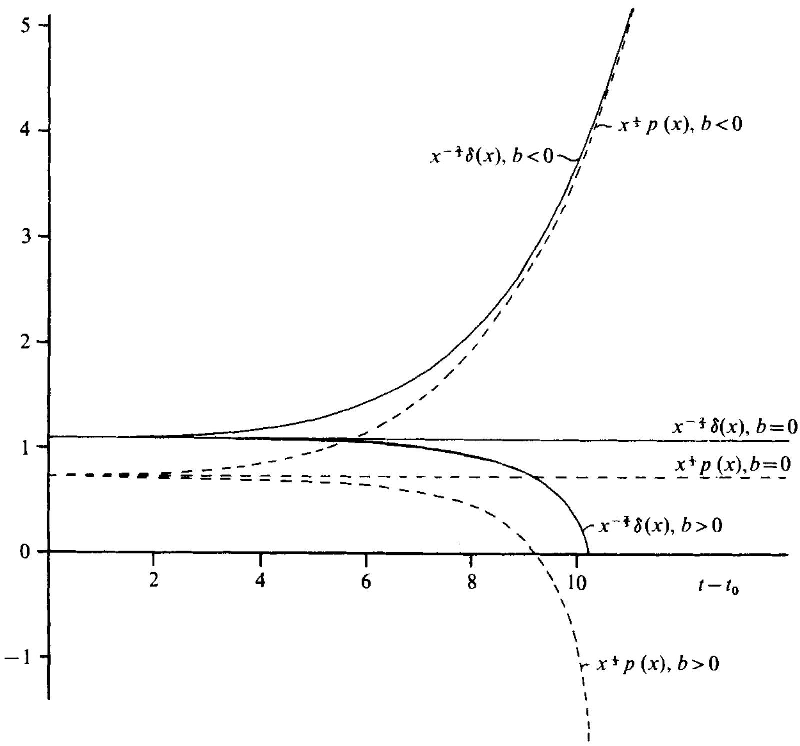

FIGURE 3. Solution branches for $b < 0$, $b = 0$, $b > 0$; $\alpha = 0{\cdot}85$.

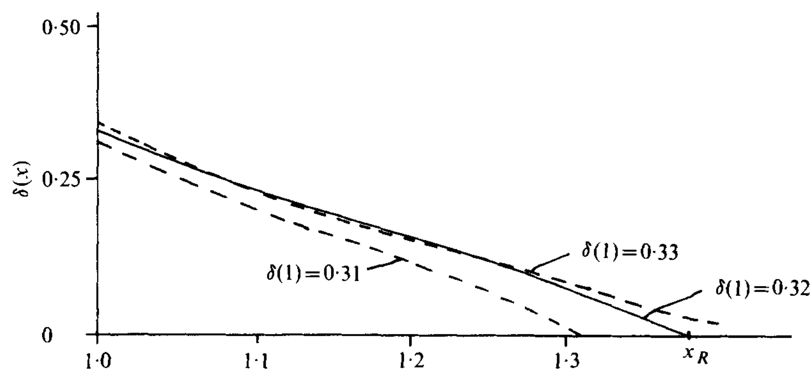

FIGURE 4. Solutions past cut-off for different values of $\delta(1)$; $\alpha = 0{\cdot}85$.

is not zero this implies that $d\theta/dt \to -\infty$ also as $t \to t_R$. In $t_0 < t < t_R$ we change the step length to a fraction of $t_R - t_0$ so that the reattachment does not occur in the middle of a step. If $t_R - t_0 \geqslant 1$, the step length is taken to be close to its value in $t < t_0$ and if $t_R - t_0 \leqslant 1$ it is taken to be $\frac{1}{10}(t_R - t_0)$.

If the value we choose for $\theta(t_0)$ is too high, the slope of θ in $t_0 < t < t_R$ is too small and $\theta(t_R)$ is finite (i.e. $\delta(x_R) \neq 0$), but if the value of $\theta(t_0)$ is too low, the slope of θ is too large and $\theta \to -\infty$ before $t = t_R$. Once $d\theta/dt$ becomes large and negative (< -100) as e^θ approaches zero, the numerical solution converges slowly and the solution for

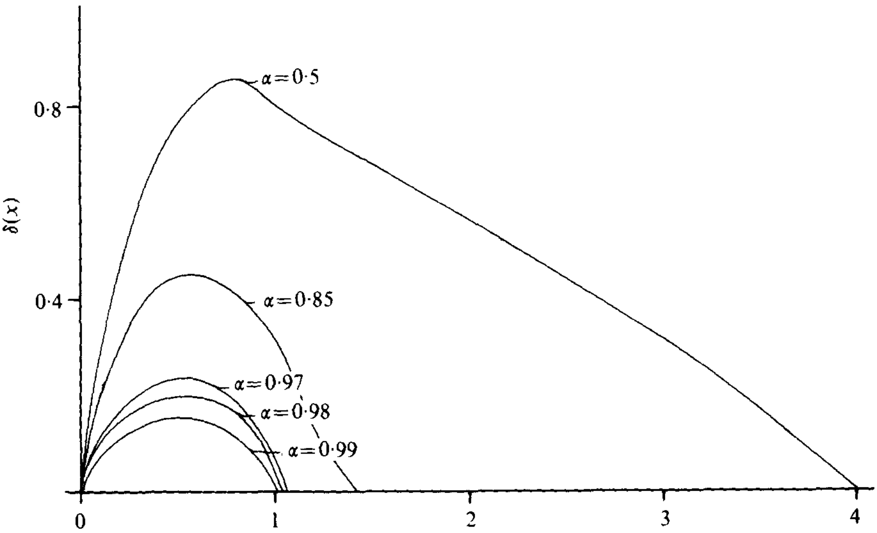

FIGURE 5. Height of injectant layer.

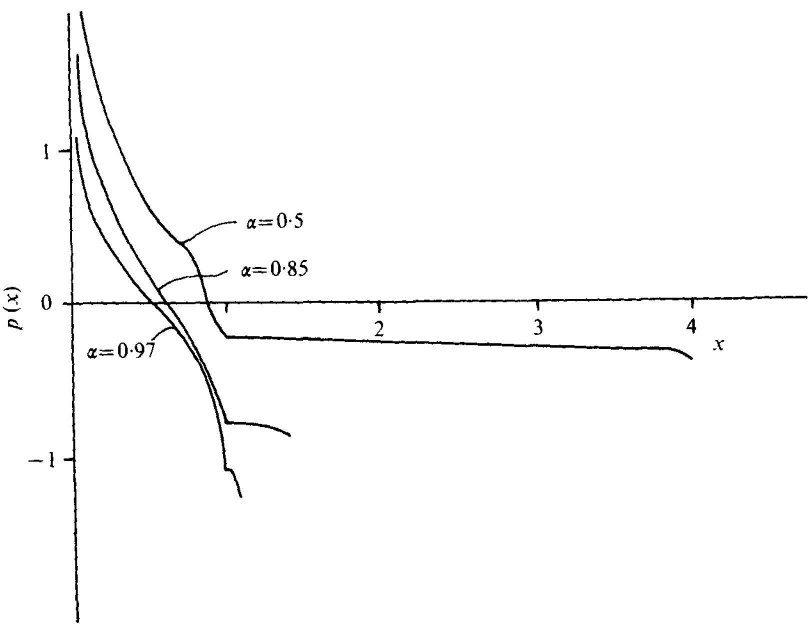

FIGURE 6. Pressure between separation and reattachment.

the last few steps is in fact found by fitting the solution to the known analytic form, in which $\delta(x)$ is linear near reattachment. We believe that the value of $\theta(t_0)$ can be found correct to three significant figures. Examples of the solution in $t_0 < t < t_R$ for a range of values of $\theta(t_0)$ are shown in figure 4 for $\alpha = 0{\cdot}85$.

By fixing $\theta(t_0)$ and therefore t_0, the distance from the initial position to cut-off, we

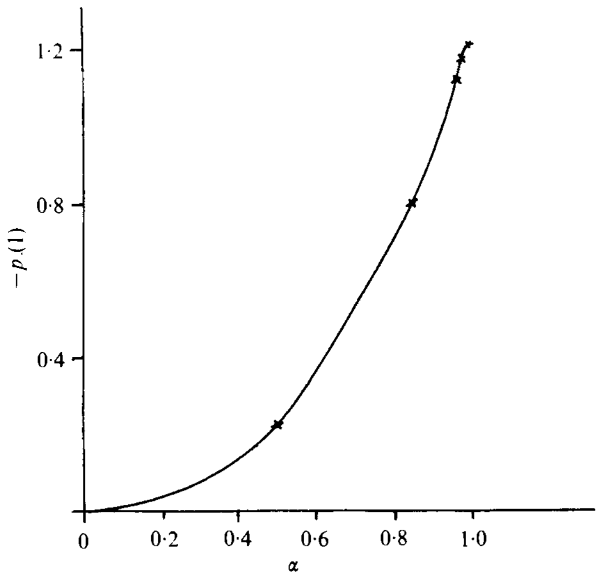

FIGURE 7. Pressure at cut-off *vs.* α.

have determined b and so matched the solution in $0 < x < 1$ with the solution in $1 < x < x_R$. Equation (4.15) gives

$$b = \frac{3}{2}a\left(\frac{d\theta}{dt}\right)_{t=0} \exp\left[\left(m-\tfrac{2}{3}\right)t_0\right]. \tag{4.16}$$

In addition to the solutions for $\alpha = 0{\cdot}99$, $0{\cdot}98$ and $0{\cdot}97$ mentioned in § 3, solutions have been found for $\alpha = 0{\cdot}85$ and $0{\cdot}5$ and a representative sample is shown in figures 5 and 6. We see that in all cases there is a discontinuity in $p'(x)$ in conformity with (2.15) and that downstream of cut-off the pressure continues to decrease to a minimum at $x = x_R-$. This last result follows from rewriting (2.8) in the form

$$v = -p'(x)\,u\int_0^v \frac{dy}{u^2} \tag{4.17}$$

and recalling that $v > 0$ at $y = \delta(x)$. As α decreases to zero, the values of $p(1)$ and $p(x_R)$ increase and seem to be approaching zero from below (see figure 7). They must in fact both be negative since the injectant fluid remaining in the inviscid layer is continually being accelerated in $1 \leqslant x < x_R$. Hence as no extra fluid is supplied the value of δ must decrease, i.e. $p < 0$ in $x \geqslant 1$. It then follows from (4.5) and the fact that $x_R - 1 = \alpha^{-2} - 1$ that as $\alpha \to 0$, $p(1)/\alpha^2$ is bounded while $p(x_R)/\alpha^{\frac{4}{3}} \to 2^{-\frac{2}{3}}$. Thus the general principles covering the cut-off and reattachment conditions formulated by Smith & Stewartson (1973*a*) for strong injection in which $V_w^* = O(R^{-\frac{3}{8}}U_\infty^*)$, or in this case $\alpha = O(R^{-\frac{1}{8}})$, are consistent with the solutions obtained here. For example, the pressure rise in strong injection is $O(R^{-\frac{1}{4}})$ and at cut-off it is also $O(R^{-\frac{1}{4}})$. According to our theory, when $\alpha \ll 1$ the pressure rise is generally $O(R^{-\frac{1}{3}}\alpha^{-\frac{2}{3}})$, at cut-off it is $O(R^{-\frac{1}{3}}\alpha^{\frac{4}{3}})$ and at reattachment it is $O(R^{-\frac{1}{3}}\alpha^{\frac{2}{3}})$. Although Smith & Stewartson's solution was not continued to reattachment, the form of the solution proposed for the region downstream of cut-off has $\delta'(x) \ll 1$ as has ours when $\alpha \ll 1$. Strictly the injection velocities in the two studies have different forms but they can easily be generalized without altering the general structure of the flow field. The only essential requirement is that in moderate blowing separation occurs at the leading edge.

5. Discussion

The key to the completion of a satisfactory description of the separated boundary layer in a supersonic flow due to a similarity-type injection velocity is the realization that even though the governing equations are either hyperbolic or parabolic the satisfaction of initial and boundary conditions in the injectant region does not establish uniqueness of the solution. There is an arbitrary constant available which enables us to satisfy the condition at reattachment that the boundary-layer thickness is reduced to the same order of magnitude as in classical theory ($\sim R^{-\frac{1}{2}}$). The solution may be generalized if required to include any moderate injection velocity ($\sim R^{-\frac{1}{2}}$) provided that it is strong enough near the leading edge to provoke separation immediately and also that sufficient fluid is injected that the free shear layer is unable to entrain it all before cut-off. The numerical procedure is similar to that used here, but the simplification of the universal solution which reduced the labour in §4 is lost with other injection velocities. Instead, it will be necessary to establish t_0 initially and carry the integration right through from the leading edge for a variety of such values until one is found which leads to δ being zero at reattachment.

Some questions of detail are however left unanswered in this paper. The actual mode of separation near the leading edge presumably has much in common with the free-interaction separation discussed by Stewartson & Williams (1969) and by Smith & Stewartson (1973*a*) but needs further study to determine, for example, the maximum pressure at the onset of blowing. Again, at cut-off δ and p are properly continuous but the pressure gradient is discontinuous and presumably a transition region is needed here to smooth it out. Finally there is a finite pressure rise at reattachment which manifests itself as a discontinuity on the scale of the injectant region. Such a phenomenon is not new in supersonic boundary-layer theory and has been described by Messiter *et al.* (1973) and by Burggraf (1973, 1975). The jump in pressure at reattachment is necessary in those problems to turn fluid entrained by the shear layer round to complete its recirculation path. Here the jump is weaker and its role is merely to reorient the free shear layer to be parallel to the plate, the circulation near reattachment being of negligible proportions. A full study of these transition regions requires an investigation on the lines of the work done by Stewartson (1974*b*) for strong injection, which, while beyond the scope of this paper, is not expected to disturb the conclusions here in a significant way.

Let us consider the changes in the structure of the boundary layer when the injection rate is gradually increased, being always of the Iglisch & Gröhne type near the leading edge and such that separation occurs there if at all. For weak blowing rates the boundary layer is only slightly thickened and the pressure rise remains $O(R^{-\frac{1}{2}})$ over the plate. Eventually a critical injection rate is reached at which separation occurs at the leading edge. At marginal blow-off rates the boundary layer begins to show signs of thickening owing to the formation of an inviscid layer of injected fluid near the plate. The most noticeable result is a sharp drop in the pressure near cut-off to a minimum value $O(R^{-\frac{1}{8}})$ below p_∞^*. The reason is that the free shear layer returns rather abruptly to the plate just after cut-off once it has entrained all the injectant. The pressure variation then ensures that the shear layer becomes parallel to the plate again. As the injection rate increases, the pressure fall at cut-off and reattachment spreads upstream and diminishes in magnitude. The pressure gradient upstream of reattachment of course remains favourable. When a state of strong injection $O(R^{-\frac{3}{8}})$ is reached the thickness of the inviscid layer reaches $O(R^{-\frac{1}{4}})$ but downstream of cut-off the pressure fall is considerably smaller in amplitude $[O(R^{-\frac{1}{2}})]$, being slightly greater $[O(R^{-\frac{5}{12}})]$ near reattachment, which occurs at a distance $O(R^{\frac{1}{4}})$ further downstream.

Other problems which may prove tractable using the non-uniqueness property found here include subsonic boundary layers with separation at the leading edge and boundary layers subject to blowing which separate a finite distance downstream of the start of injection. The occurrence of separation in the latter cases is probably similar to that described by Catherall *et al.* (1965) and the free shear layer which develops downstream cannot be expected to be immediately of the self-similar type investigated by Iglisch & Gröhne (1945). Further discussion of this problem is postponed.

Some numerical results for blowing with a finite Reynolds number have recently become available (Werle 1977). In form, these compare well with the slot blowing studies of Smith & Stewartson (1973*b*) and by implication support the results of this paper. The injection distributions which Werle considers are however sufficiently unlike those that we use for us to be unable to make any direct comparisons at present.

REFERENCES

AMR, A. & KASSOY, D. R. 1973 *Phys. Fluids* **16**, 1579–1586.

AMR, A. & KASSOY, D. R. 1976 *Phys. Fluids* **19**, 1312–1318.

BROWN, S. N. & STEWARTSON, K. 1969 *Ann. Rev. Fluid Mech.* **1**, 45–72.

BURGGRAF, O. R. 1970 *U.S. Air Force Aerospace Res. Lab. Rep.* ARL 70-0275.

BURGGRAF, O. R. 1973 *Proc. 3rd Int. Conf. Num. Methods in Fluid Mech., Lecture Notes in Physics*, vol. 18. Springer.

BURGGRAF, O. R. 1975 *AGARD Symp. Separation, Göttingen*, paper 10.

CATHERALL, D., STEWARTSON, K. & WILLIAMS, P. G. 1965 *Proc. Roy. Soc.* A **284**, 270–296.

CEBECI, T. & BRADSHAW, P. 1977 *Momentum Transfer in Boundary Layers.* Hemisphere, McGraw-Hill.

CHANG, C. J. & MESSITER, A. F. 1968 *Univ. Michigan Willow Run Lab. Rep.* no. 8430-2-T.

COLE, J. D. & AROESTY, J. 1968 *Int. J. Heat Mass Transfer* **11**, 1167–1183.

DIVER, C. 1978 Ph.D. thesis, University College London (in preparation).

EMMONS, H. W. & LEIGH, D. C. 1953 *Aero. Res. Counc. Current Paper* no. 157.

GADD, G. E., JONES, C. W. & WATSON, E. J. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead), chap. VI. Oxford University Press.

IGLISCH, R. & GRÖHNE, D. 1945 *Ber. Inst. Math. Tech. Hochsch., Braunschweig*, no. 1–45.

KASSOY, D. R. 1974 *J. Fluid Mech.* **62**, 145–161.

KELLER, H. B. & CEBECI, T. 1971 *2nd Int. Conf. Num. Math. Fluid Dyn., Berkeley*, p. 92. Springer.

KLEMP, J. B. & ACRIVOS, A. 1972 *J. Fluid Mech.* **51**, 337–356.

MESSITER, A. F., HOUGH, G. R. & FEO, A. 1973 *J. Fluid Mech.* **60**, 605–624.

PRETSCH, J. 1944 *Unter Mitt. Deut. Luftfahrt Rep.* no. 3091.

SCHLICHTING, H. & BUSSMAN, K. 1943 *Schr. Dtsch. Akad. Luftfahr*, no. 7B.

SMITH, F. T. & STEWARTSON, K. 1973*a* *J. Fluid Mech.* **58**, 143–159.

SMITH, F. T. & STEWARTSON, K. 1973*b* *Proc. Roy. Soc.* A **332**, 1–22.

Stewartson, K. 1974*a* *Adv. Appl. Math.* **14**, 146–239.
Stewartson, K. 1974*b* *J. Fluid Mech.* **62**, 289–308.
Stewartson, K. & Williams, P. G. 1969 *Proc. Roy. Soc.* A **312**, 181–206.
Werle, M. J. 1977 *Dept. Aerospace Engng Appl. Math. Univ. Cincinnati, Rep.* AFL 77-10-35.